Stabilized Virtual Element Method for the nonlinear convection-diffusion problems

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by

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Abstract

This work concerns residual-based stabilization of the Virtual Element Method for non-linear convection-diffusion problems. It is well-known that the numerical simulations of singularly perturbed problem produce solutions with spurious oscillations. In chapter one, we discuss the Galerkin approximation of the convection-diffusion equation. From the investigation of a simple one-dimensional problem, it is revealed that there is an onset of unphysical oscillations in the Galerkin solution for dominant convection. From one perspective, very rigorous mesh refinement acts as a remedy. As this resolve is non-viable, we study residual-based stabilization methods that circumvent mesh fine-tuning. Then we introduce the polytopal Galerkin method called the Virtual Element Method. We clearly state the advantage of VEM over the existing polytopal methods and briefly give the construction of the VEM space. We demonstrate the usage of the polynomial projection operators Π_p^{∇}, Π_p^0 and $\mathbf{\Pi}_{p-1}^0$.

Chapter two is devoted to studying the SUPG stabilization of VEM for the semilinear convection-diffusion-reaction equation. We prove theoretical estimates involving the mesh size h and the polynomial order p. For analysis, we prove the existence of an interpolation operator onto VEM space with optimal approximation property with respect to both the parameters $h,\ p$ for L^2 norm and H^1 semi-norm. Under suitable choice of the SUPG parameter, the error estimate showing optimal order of convergence is derived. We obtain the optimal convergence rate in H^1 semi-norm and L^2 norm for convection-dominated and reaction-dominated phenomena, respectively. In fact we obtain optimal order for the energy norm $|||\cdot|||$. Numerical experiments conducted verified our theoretical results over convex and nonconvex meshes for VEM order $p = 1, 2, 3$.

The shock-capturing stabilization of VEM for the convection-diffusion equation is analyzed in chapter 3. We begin by formulating a computable VEM scheme stabilized with the shock-capturing technique for the linear convection-diffusion-reaction equation. It is noted that the discretization of a linear problem produced a nonlinear discrete scheme. The existence of the VEM solution was shown with the help of a variant of Brouwers fixed point theorem. The efficiency of the shock-capturing method was investigated numerically by comparing it with the SUPG method, for a linear problem with discontinuous boundary conditions, on different polygonal meshes. With the success of shock-capturing in reducing

spurious oscillations, we proceed to investigate in detail the shock-capturing stabilization of VEM for the semilinear convection-diffusion equation. We discussed two variants of shock-capturing technique, where in the first case, we add isotropic artificial diffusion, and the second type adds anisotropic diffusion. Error estimate with similar order of convergence as the SUPG method is derived. We used the Newton method in the simulations to solve a nonlinear system. Numerical experiments conducted reveal the effectiveness of the shock-capturing stabilization in diminishing the cross-wind oscillations present in the SUPG solution.

The fourth chapter discusses the SUPG stabilization of VEM for the quasilinear convection-diffusion-reaction equation. In this, we study the approximation of branches of nonsingular solutions. We show the existence and uniqueness of a branch of discrete solution approximating the branch of the nonsingular solution through results proved by Brezzi et al. for a much general class of nonlinear equations. Convergence estimate showing optimal order for H^1 seminorm and the energy norm $|||\cdot|||$ were derived. Since the problem is quasilinear, on the fine mesh using the Newton method to solve the system is time-consuming. Therefore we use the two-grid method that involves two meshes of different mesh sizes for solving the nonlinear system of equations. Numerical experiments conducted verified the theoretical results. The CPU time taken by the two-grid for solving the system is halved compared to the time taken by the Newton method on a fine mesh.

In Chapter 5, we consider the discretization of the nonlocal coupled parabolic problem within the framework of the virtual element method. In fully discrete formulation, the backward Euler method is used for discretizing the time derivative, and VEM is used for spatial discretization. The presence of nonlocal coefficients makes the computation of the Jacobian more expensive in Newton's method and destroys the sparsity of the Jacobian. In order to resolve this problem, we propose an equivalent formulation that yields a sparse Jacobian. We derive the error estimates in the L^2 and H^1 norms. A linearised scheme without compromising the convergence rate in different norms is proposed to reduce the computational complexity further. Finally, the theoretical results are verified through the numerical experiments conducted on arbitrary polygonal meshes.

The final chapter discusses the possible works related to problems studied in this thesis that can be investigated in the future.